A Reasonably Gradual Dependent Type Theory

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Monday, 12th of September, 2022

Prototyping with dependent types


```
Inductive vect (A : Type) : ℕ → Type :=
| nil : vect A 0
| _::_ {n : ℕ} (head : A) (tail : vect A n) : vect A (1+n)
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vect A (1+ ?)

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Prototyping with dependent types


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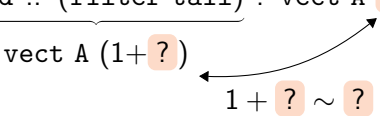

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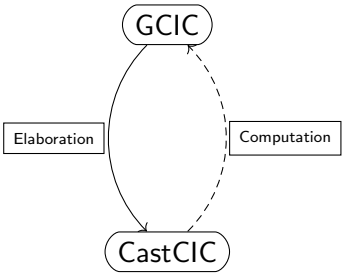
`vect A (1+?)`

`1 + ? ~ ?`

```
if l = nil
then 0
else ?
```

GCIC & CastCIC: a marriage of Gradual & Dependent Types



- Unknown terms and types $?_A : A$
- Errors $\texttt{err}_A : A$
- Casts $t : A \implies \langle B \Leftarrow A \rangle\, t : B$

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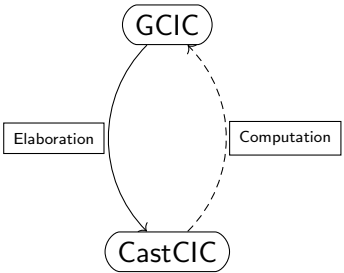

Precision between types

$\mathbb{N} \to \mathbb{N} \sqsubseteq \mathbb{N} \to ? \sqsubseteq ? \to ? \sqsubseteq ?$

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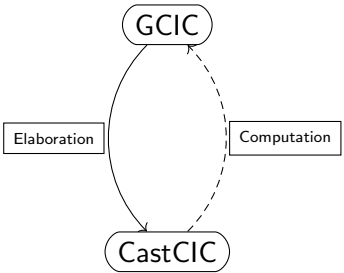

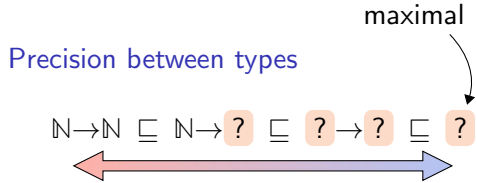

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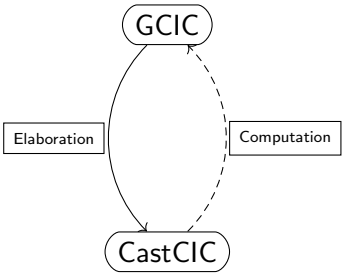

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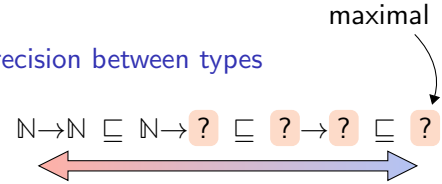


Precision enforces cast validity:

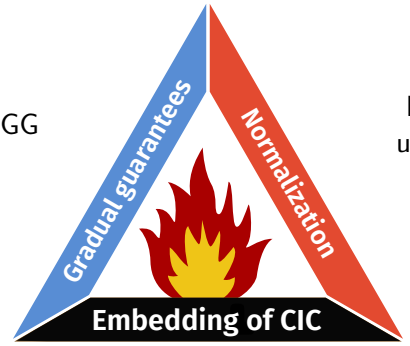
$$A \sqsubseteq B \implies A \underset{\langle A \Leftarrow B \rangle}{\overset{\langle B \Leftarrow A \rangle}{\rightleftarrows}} B$$

Graduality [New & Ahmed 2018]

The Fire Triangle of Graduality

From Gradualizing the CIC [TOPLAS 2022]

Graduality
Static & Dynamic GG



$?_\square \to ?_\square \not\sqsubseteq ?_\square$
No embedding of
untyped λ-calculus

Conservativity: t, A ? -free

t has type A in GCIC iff t has type A in CIC

$\vdash_? t : A \iff \vdash_{\text{CIC}} t : A$

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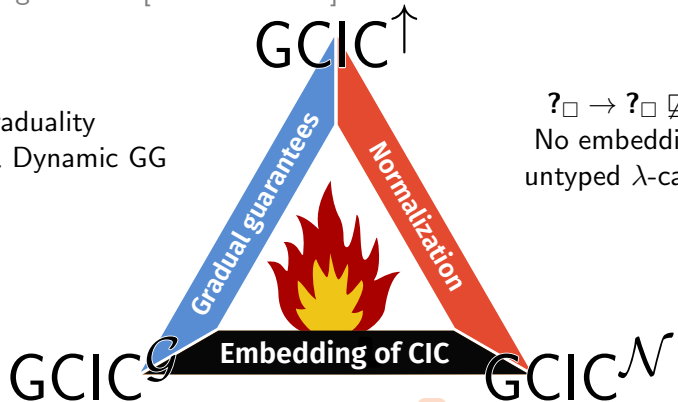

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Internal precision in GRIP

$\mathsf{CastCIC}^{\mathcal{N}}$: Normalizing, conservative over CIC, not gradual

How far is $\mathsf{CastCIC}^{\mathcal{N}}$ from actually being gradual ?

Internal precision in GRIP

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How far is $\mathsf{CastCIC}^{\mathcal{N}}$ from actually being gradual ?

Make precision internal so that we can reason on it!

$$\mathsf{GRIP} = \mathsf{CastCIC}^{\mathcal{N}} + \sqsubseteq + \mathbb{P}$$

Internal precision in GRIP

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$$\mathsf{GRIP} = \mathsf{CastCIC}^{\mathcal{N}} + \sqsubseteq + \mathbb{P}$$

$$\frac{\Gamma \vdash A, B : \square_i}{\Gamma \vdash A \sqsubseteq_i B : \mathbb{P}} \qquad \frac{\Gamma \vdash A, B : \square_i \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash a \ {}_A\sqsubseteq_B \ b : \mathbb{P}}$$

$\mathbb{P}$: Pure universe of propositions

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$\mathbb{P}$: Pure universe of propositions for sound reasoning

- Pure: no error, no casts, no ?
- propositions: definitionally proof-irrelevant

Proofs of precision – External meaning

Dynamic Gradual Guarantee

$$\texttt{DGG} : \forall (A : \square)(C : A \to \mathbb{B})(x\ y : A), x\ {}_A\sqsubseteq_A\ y \to C\ x\ {}_{\mathbb{B}}\sqsubseteq_{\mathbb{B}}\ C\ y.$$

$\texttt{DGG}\ A\ C$ holds $\iff$ $C\ {}_{A\to\mathbb{B}}\sqsubseteq_{A\to\mathbb{B}}\ C$ $\iff$ C monotone wrt. $\sqsubseteq$

Proofs of precision – External meaning

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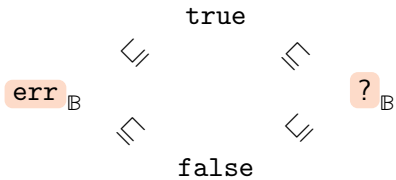
$$\texttt{DGG}\ A\ C \text{ holds} \iff C\ {}_{A\to\mathbb{B}}\sqsubseteq_{A\to\mathbb{B}}\ C \iff C \text{ monotone wrt. } \sqsubseteq$$

DGG correspond to error-approximation:



Proofs of precision – An example

$$\texttt{add1} : \mathbb{N} \to \mathbb{N} := \lambda x : \mathbb{N}.x + 1$$
$$\texttt{add?} : \textbf{?}_{\square} \to \mathbb{N} := \lambda x : \textbf{?}_{\square}.(\langle \mathbb{N} \Leftarrow \textbf{?}_{\square} \rangle\, x) + 1$$

$\texttt{add1} \;{}_{\mathbb{N}\to\mathbb{N}}\!\sqsubseteq_{\textbf{?}_{\square}\to\mathbb{N}} \texttt{add?} \quad \Longrightarrow \quad$ add? does not error on good input

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Proofs of precision – Principles

Reasoning principles for $\sqsubseteq$

- heterogeneous transitivity

$$a \ {}_A\sqsubseteq_B \ b \wedge b \ {}_B\sqsubseteq_C \ c \to a \ {}_A\sqsubseteq_C \ c,$$

- quasi-reflexivity

$$a \ {}_A\sqsubseteq_B \ b \to a \ {}_A\sqsubseteq_A \ a \wedge b \ {}_B\sqsubseteq_B \ b,$$

- adjunction properties

$$A \sqsubseteq_i B \to \langle B \Leftarrow A \rangle \, a \ {}_B\sqsubseteq_B \ b \iff a \ {}_A\sqsubseteq_B \ b \iff a \ {}_A\sqsubseteq_A \ \langle A \Leftarrow B \rangle \, b,$$

- decomposition through upper bounds

$$A, B \sqsubseteq_i X \to a \ {}_A\sqsubseteq_B \ b \iff \langle X \Leftarrow A \rangle \, a \ {}_X\sqsubseteq_X \ \langle X \Leftarrow B \rangle \, b.$$

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- $?_{\square_i} \to ?_{\square_i} \not\sqsubseteq_i ?_{\square_i}$
- $?_{\square_i}$ maximal for ${}_{\square_i}\sqsubseteq_{\square_i}$, not for $\sqsubseteq_i$

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$$A, B \sqsubseteq_i X \to a \ {}_A\sqsubseteq_B \ b \iff \langle X \Leftarrow A \rangle \, a \ {}_X\sqsubseteq_X \langle X \Leftarrow B \rangle \, b.$$

However

- $A \ {}_{\Box_i}\sqsubseteq_{\Box_i} \ B \iff A \sqsubseteq_i B \wedge B \sqsubseteq_i ?_{\Box_i}$
- $?_{\Box_i} \to ?_{\Box_i} \not\sqsubseteq_i ?_{\Box_i}$
- $?_{\Box_i}$ maximal for ${}_{\Box_i}\sqsubseteq_{\Box_i}$, not for $\sqsubseteq_i$
- Mitigated by explicity cumulativity

$$\iota(?_{\Box_i} \to ?_{\Box_i}) \sqsubseteq_{i+1} ?_{\Box_{i+1}}$$

How good is GRIP? Metatheory!

Consistency $\not\vdash_{\mathsf{GRIP}} t : \bot$ (where $\bot : \mathbb{P}$)

Idea: Build a model of GRIP in MLTT+SProp using partial pre-orders.
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Gradual fragment $\mathsf{GRIP}^{\uparrow}$ Automatically precise terms

Idea: Raise universe levels on Π

$$\frac{\Gamma \vdash_{\mathsf{GRIP}^{\uparrow}} A : \square_i \qquad \Gamma, x : A \vdash_{\mathsf{GRIP}^{\uparrow}} B : \square_i}{\Gamma \vdash_{\mathsf{GRIP}^{\uparrow}} \Pi\, x : A.B : \square_{i+1}}$$

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GRIP = Partially gradual type theory + internal precision

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GRIP is a type theory, not yet a proof assistant:

- ▷ Elaboration from a gradual source
- ▷ General inductive types
- ▷ An actual implementation?

Thank You!

Gradualizing printf


```
Equations argOfDescr (s : stringDescr) : Type :=
  argOfDescr "" := unit ;
  argOfDescr ("%d" @ rest) := ℕ → argOfDescr rest;
  ...

Definition printf : forall (s : stringDescr), argOfDescr s := ...
```

- ▷ `printf` does not type in GRIP$^{\uparrow}$ (arbitrary arity)
- ▷ `printf` does type in GRIP
- ▷ It is not self-precise
- ▷ but its retrictions to `stringDescr` bounded by some fixed length are self-precise.